Classical bits: 0 or 1

Quantum bits: $|0\rangle$ or $|1\rangle$
(qubit) "ket 0" "ket 1"

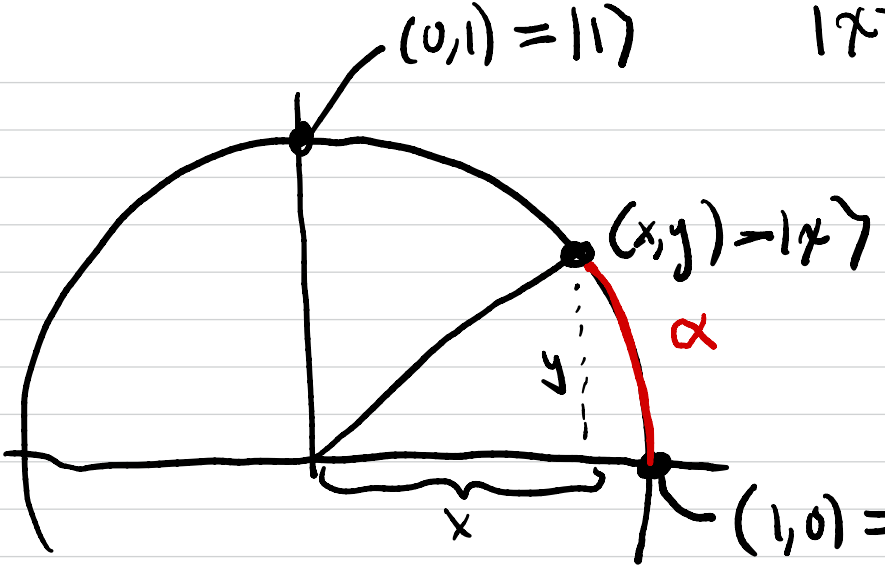
or "superposition" of $|0\rangle$ and $|1\rangle$

$$|\psi\rangle = x \cdot |0\rangle + y \cdot |1\rangle \qquad x, y \in \mathbb{R} \qquad x^2 + y^2 = 1$$

(actually can be complex...)

e.g. $|+\rangle = \frac{1}{\sqrt{2}} \cdot |0\rangle + \frac{1}{\sqrt{2}} \cdot |1\rangle$

$|-\rangle = \frac{1}{\sqrt{2}} \cdot |0\rangle - \frac{1}{\sqrt{2}} \cdot |1\rangle$



$$|x\rangle = x\cdot|0\rangle + y\cdot|1\rangle$$
$$= x\cdot(1,0) + y\cdot(0,1)$$
$$= (x,0) + (0,y)$$
$$= (x,y)$$

$$|\chi\rangle = \cos(\alpha)\cdot|0\rangle + \sin(\alpha)\cdot|1\rangle$$

e.g. $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \cos\frac{\pi}{4}\cdot|0\rangle + \sin\frac{\pi}{4}\cdot|1\rangle$

Quantum Measurements

Given $|\psi\rangle = x\cdot|0\rangle + y\cdot|1\rangle$ $\quad (x^2+y^2=1)$
we can "measure" it

w/prob x^2: "observe" 0
$|\psi\rangle$ becomes $|0\rangle$ "collapse of the wave function"

w/prob y^2: "observe" 1
$|\psi\rangle$ becomes $|1\rangle$

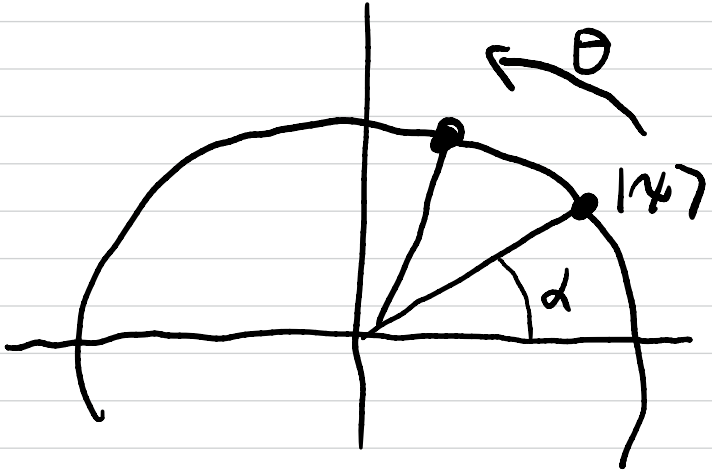
e.g. measure $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$, observe 0 w/prob $\frac{1}{2}$
observe 1 w/prob $\frac{1}{2}$

measure $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$, observe 0 w/prob $\frac{1}{2}$
observe 1 w/prob $\frac{1}{2}$

measure $|0\rangle$ always observe 0

Quantum operations

One operation: Rotate(θ) ← parameter

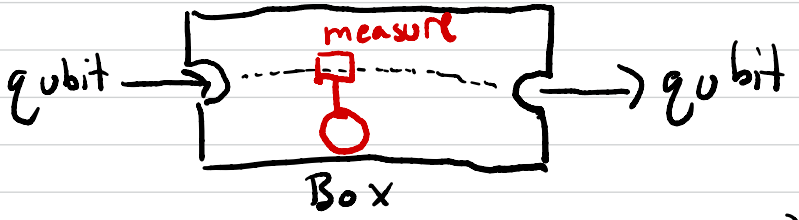


If $|\psi\rangle = \cos(\alpha)\,|0\rangle + \sin(\alpha)\,|1\rangle$

moves it to

$\cos(\alpha+\theta)\,|0\rangle + \sin(\alpha+\theta)\,|1\rangle$

Elitzur-Vaidman Bomb Algorithm



Q: Is there a bomb?

Box is either (1) empty or (2) contains a bomb

If (1) empty, and we send $|x\rangle = x|0\rangle + y|1\rangle$
nothing happens & $|x\rangle$ comes out on other end

If (2) bomb, if measures $|x\rangle$
If observes 0, nothing happens
($|x\rangle = |0\rangle$ comes out other end)

If observes 1, bomb explodes

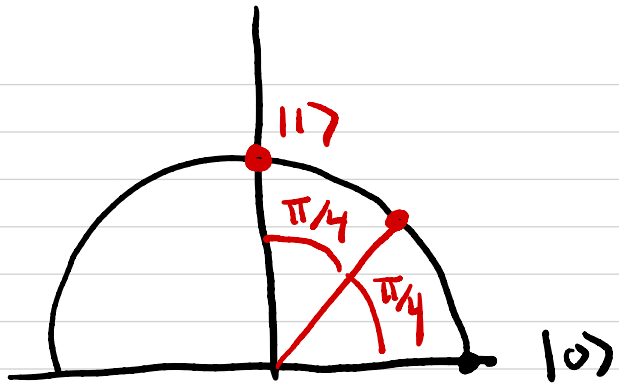
Classical strategy

If we send in 107,
no info

If we send in 117
and box has bomb,
explosion!

Quantum strategy

$|\psi\rangle = |0\rangle$
Rotate($\pi/4$, $|\psi\rangle$) (45°)
Send $|\psi\rangle$ through box
Rotate($\pi/4$, $|\psi\rangle$)
Measure $|\psi\rangle$
If observe 0, output "Bomb!"
If observe 1, output "Not sure"



If no bomb: $|0\rangle \xrightarrow[\pi/4]{\text{rotate}} \cos(\pi/4)|0\rangle + \sin(\pi/4)|1\rangle \; (= |+\rangle)$

$\rightarrow$ [empty] $\rightarrow |\psi\rangle = \cos(\pi/4)|0\rangle + \sin(\pi/4)|1\rangle$

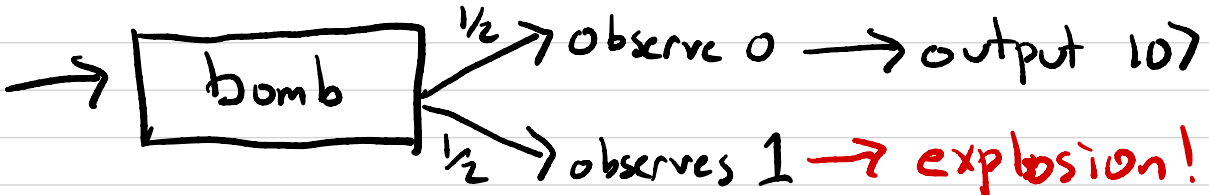
$\xrightarrow[\pi/4]{\text{rotate}} \cos(\pi/2)|0\rangle + \sin(\pi/2)|1\rangle$

$= 0 \cdot |0\rangle + 1 \cdot |1\rangle$

$\therefore$ Always output "Not sure!"

If bomb

$|0\rangle \xrightarrow[\pi/4]{\text{rotate}} \cos(\tfrac{\pi}{4})|0\rangle + \sin(\tfrac{\pi}{4})|1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$



In first case: $|0\rangle \xrightarrow[\pi/4]{\text{rotate}} \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ → 1/2 observe 0 "Bomb!"; 1/2 observe 1 "Not sure!"

w/prob 1/2 1/4 1/4

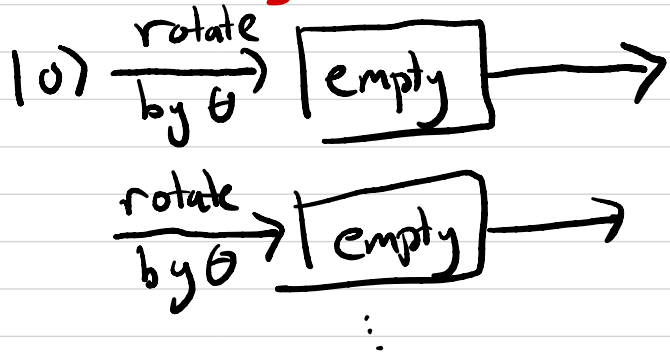
explosion "Bomb!" "Not sure"

N = Input("Safety level?")
$|\chi\rangle = |0\rangle$
$\theta = (\pi/2)/N$

for $t = 1 \ldots N$
  Rotate(θ, $|\chi\rangle$)
  Send $|\chi\rangle$ through box

Measure $|\chi\rangle$
If observe 0, output "Bomb"
If observe 1, output "Empty"

If empty



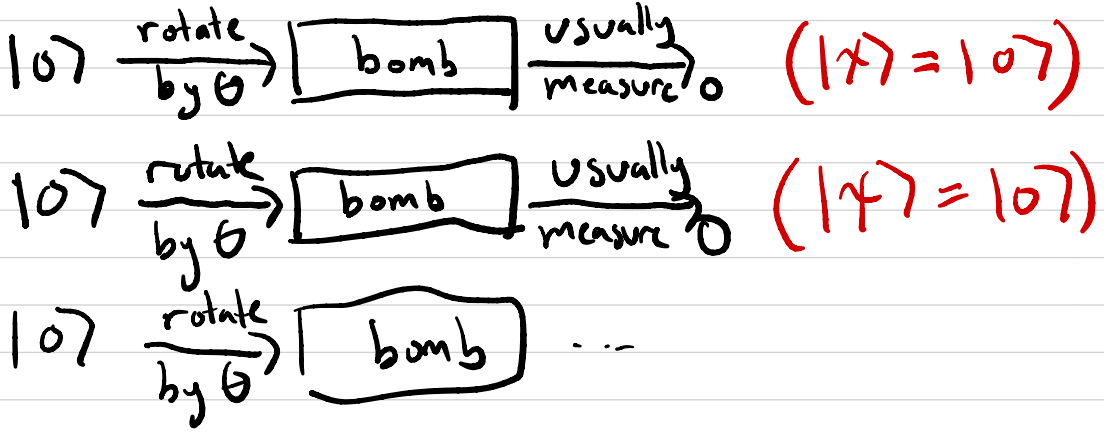
rotate by θ N times
= rotate by $N \cdot \theta = \pi/2$

At end $|\chi\rangle = \cos(\frac{\pi}{2})|0\rangle + \sin(\frac{\pi}{2})|1\rangle$

$= |1\rangle$

$\therefore$ Always see "Empty"

If contains bomb



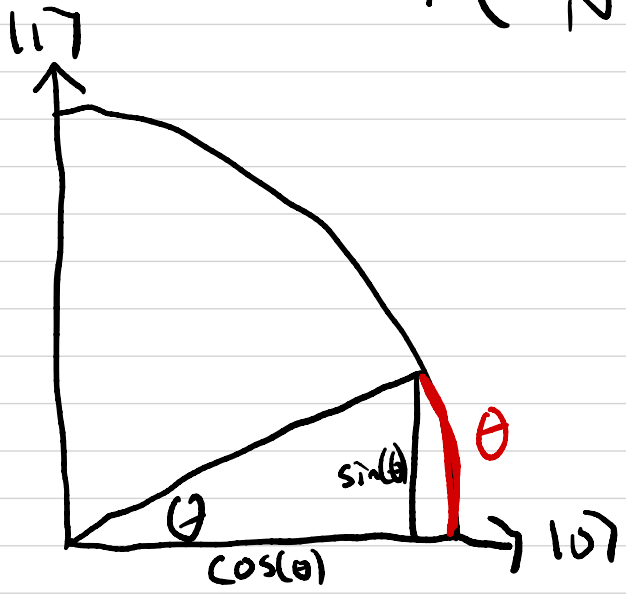
If no explosion, final state is $|0\rangle$ $\therefore$ output "Bomb!"

$$\Pr[\text{explosion}] = \Pr[\text{explosion at time 1}] + \Pr[\text{explosion at time 2}] + \cdots$$

$$\Pr[\text{explosion at time } i] \leq \Pr[\underbrace{\cos(\theta)\cdot|0\rangle + \sin(\theta)\cdot|1\rangle}_{\text{measures to 1}}] = \sin(\theta)^2$$

$$\Pr[\text{explosion}] \leq N \cdot \sin(\theta)^2$$

$$\Pr[\text{explosion}] \le N \cdot (\sin\theta)^2 \le N \cdot \theta^2$$
$$= N\left(\frac{\pi/2}{N}\right)^2 = \frac{(\pi/2)^2}{N} \le \frac{2.5}{N}$$



Summary

If empty, 100% chance to output "empty"

If bomb, explosion w/prob $2.5/N$

otherwise, output "bomb!"

Uses $O(N)$ time

↑ Safety level