Multiplicative Weights

Applications

Online algorithms

There are n experts/actions and T days.

On each day $t = 1 \ldots T$:

1. You specify a distribution $w^{(t)} = (w_1^{(t)}, \ldots, w_n^{(t)})$ over experts.
2. You observe costs $c^{(t)} = (c_1^{(t)}, \ldots, c_n^{(t)})$ in ~~$[0,1]$~~ $[a,b]$ range $= b-a$

$$\therefore \ \mathbb{E}[\text{Your cost}] = w_1^{(t)} \cdot c_1^{(t)} + w_2^{(t)} \cdot c_2^{(t)} + \cdots = w^{(t)} \cdot c^{(t)}$$

Thm: If you run the Multiplicative Weights Alg,

(Your avg cost) $\leq$ (avg cost of best expert) + (small)

$$\frac{1}{T}\sum_{t=1}^{T} w^{(t)} \cdot c^{(t)} \leq \min_{1 \leq i \leq n} \left\{ \frac{1}{T}\sum_{t=1}^{T} c_i^{(t)} \right\} + \underbrace{\boxed{O\left(\text{range} \cdot \sqrt{\frac{\log(n)}{T}}\right)}}_{= \text{"regret"}}$$

$\therefore$ MW is a "no regret alg"

Application 1:

Minimax Theorem

Zero sum games

Let $M = \begin{bmatrix} M_{11} & & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{n1} & & M_{nn} \end{bmatrix}$ be an $n \times n$ payoff matrix, with $-1 \le M_{ij} \le 1$.

Given mixed strategies $p = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$ and $q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$,

$$\text{Score}(p,q) = \sum_{i,j} p_i M_{ij} q_j = p^T M q.$$

Minimax Thm: $\min\limits_{q} \max\limits_{p} p^T M q = \max\limits_{p} \min\limits_{q} p^T M q$

(col player 1st) (row player 1st)

Easy direction: $\ge$

Hard direction: $\le$ (via strong duality of LPs)

(or via MW)

Minimax proof

$$\begin{array}{ccc} \multicolumn{3}{c}{\text{Col player's actions}} \\ \text{Col } 1 & \cdots & \text{Col } n \\ \downarrow & & \downarrow \\ \multicolumn{3}{c}{\left[\begin{array}{c} \text{Payoff} \\ \text{Matrix} \\ M \end{array}\right]} \end{array}$$

For $t=1,\dots,T$ rounds:

1. Col player selects mixed strat $q^{(t)}=(q_1^{(t)},\dots,q_n^{(t)})$
   using multiplicative weights
2. Row player "best responds"
   $p^{(t)} = \text{argmax}\{(p^{(t)})^T M q^{(t)}\}$
3. Cost vector $c^{(t)}=(c_1^{(t)},\dots,c_n^{(t)}) = p^{(t)}\cdot M$ for col player is revealed.
4. Col player's expected loss $= p^{(t)} M q^{(t)}$.

Def: $\bar{p} = \sum_{t=1}^{T} p^{(t)}$ $\quad \bar{q} = \sum_{t=1}^{T} q^{(t)}$

Goal: $\bar{p}$ and $\bar{q}$ are "approx minimax strats":

$$\max_p \, p M \bar{q} \le \min_q \, \bar{p} M q + \text{small}$$

1. Col player picks $q^{(t)}$. 2. Row player sets $p^{(t)} = \max\{p M q^{(t)}\}$. 3. Cost $= p^{(t)} M$.

Fact: $\max_p p M \bar{q} \leq \min_q \bar{p} M q + O\left(\sqrt{\frac{\log(n)}{T}}\right)$

Pf:
$$\begin{aligned}
\max_p p M \bar{q} &= \max_p \frac{1}{T} \sum_{t=1}^{T} p M q^{(t)} \\
&\leq \frac{1}{T} \sum_{t=1}^{T} \max_p p M q^{(t)} \\
&= \frac{1}{T} \sum_{t=1}^{T} \boxed{p^{(t)} M q^{(t)}} \quad = \text{Alg's round } t \text{ expected loss} \\
&\leq \min_{i \in [n]} \frac{1}{T} \sum_{t=1}^{T} (p^{(t)} M)_i + O\left(\sqrt{\frac{\log(n)}{T}}\right). \\
&= \min_{i \in [n]} (\bar{p} M)_i + O\left(\sqrt{\frac{\log(n)}{T}}\right) \\
&= \min_q \bar{p} M q + O\left(\sqrt{\frac{\log(n)}{T}}\right). \quad \square
\end{aligned}$$

Pf of minimax:
$$\begin{aligned}
\min_q \max_p p M q &\leq \max_p p M \bar{q} \\
&\leq \min_q \bar{p} M q + O\left(\sqrt{\frac{\log(n)}{T}}\right) \\
&\leq \max_p \min_q p M q + \boxed{O\left(\sqrt{\frac{\log(n)}{T}}\right)} \xrightarrow{T \to 0} 0 \quad \square
\end{aligned}$$

Application 2:

Solving LPs (approximately)

Note: can guess OPT via binary search

n variables $x_1, \dots, x_n$

LP: $\max \ \cancel{c}^{\,z} \cdot x$ $\quad\rightarrow\quad$ $z \cdot x \geq OPT$

s.t. $a_1 \cdot x \leq b_1$ $\qquad$ $a_1 \cdot x \leq b_1$

$\vdots$ $\qquad$ $\vdots$

$a_m \cdot x \leq b_m$ $\qquad$ $a_m \cdot x \leq b_m$

$x_1, \dots, x_n \geq 0$ $\qquad$ $x_1, \dots, x_n \geq 0$

Fact: Easy if only 1 constraint.

$$\boxed{\begin{array}{l} c \cdot x \geq OPT \\ \\ x_1, \ldots, x_n \geq 0 \end{array}} = K$$

$$a \cdot x \leq b$$

Pf: For simplicity, assume $c \geq 0$ and $a \geq 0$ and $b \geq 0$.

1.) "Put everything on x_1": Set $x_1 = \frac{b}{a_1}$, $x_2 = \ldots = x_n = 0$. If feasible, output x.

2.) "Put everything on x_2"

$\vdots$

n+1) If all fail, output "not feasible".

LP alg $\quad K = \{z \cdot x \geq OPT, \ x_i \geq 0\}$

For $t = 1 \ldots T$ rounds

1. Select distribution over constraints $w^{(t)} = (w_1^{(t)}, \ldots, w_m^{(t)})$

   using multiplicative weights

2. Consider mixture of constraints $\underbrace{\sum_{i=1}^{m} w_i^{(t)} \cdot a_i}_{\alpha^{(t)}} \cdot x \leq \underbrace{\sum_{i=1}^{m} w_i^{(t)} \cdot b_i}_{\beta^{(t)}}$

3. Solve for point $x^{(t)}$ in

   $K \cup \{\alpha^{(t)} \cdot x \leq \beta^{(t)}\}$

4. Cost vector $c^{(t)} = (c_1^{(t)}, \ldots, c_m^{(t)})$, $\quad c_i^{(t)} = b_i - a_i \cdot x^{(t)}$.

5. Set $\bar{x} = \frac{1}{T}(x_1 + \cdots + x_T)$

Note: $z \cdot \bar{x} = \frac{1}{T}(z \cdot x^{(1)} + \cdots + z \cdot x^{(T)}) = OPT$

Note: $\bar{x}_i = \frac{1}{T}(x^{(1)} + \cdots + x^{(T)}) \geq 0$

Thm: For all $1 \leq i \leq m$, $a_i \cdot \bar{x} \leq b_i + (\text{small})$

$\alpha^{(t)} = \sum_i w_i^{(t)} \cdot a_i, \quad \beta^{(t)} = \sum_i w_i^{(t)} \cdot b_i, \quad x^{(t)} \in K \cup \{\alpha^{(t)} \cdot x \leq \beta^{(t)}\}$

cost $c_i^{(t)} = b_i - a_i \cdot x^{(t)}$

Thm: For all $1 \leq i \leq m$, $a_i \cdot \overline{x} \leq b_i + (\text{small})$

Pf: By MW Guarantee,

$$\frac{1}{T} \sum_{t=1}^{T} c^{(t)} \cdot w^{(t)} \leq \min_i \left\{ \frac{1}{T} \cdot \sum_{t=1}^{T} c_i^{(t)} \right\} + O\left(\text{range} \cdot \sqrt{\frac{\log(m)}{T}}\right)$$

LHS: $\frac{1}{T} \sum_t \sum_{i=1}^{m} w_i^{(t)} \cdot (b_i - a_i \cdot x^{(t)}) = \frac{1}{T} \sum_t \beta^{(t)} - \alpha^{(t)} \cdot x^{(t)} \geq 0$

RHS: $\frac{1}{T} \sum_t (b_i - a_i \cdot x^{(t)}) = b_i - a_i \cdot \overline{x}$

$\therefore \quad 0 \leq b_i - a_i \cdot \overline{x} + O\left(\text{range} \cdot \sqrt{\frac{\log(m)}{T}}\right)$

$\Rightarrow a_i \cdot \overline{x} \leq b_i + O\left(\text{range} \cdot \sqrt{\frac{\log(m)}{T}}\right) = b_i + \varepsilon$ if $T = O\left(\frac{\text{range}^2 \log(m)}{\varepsilon^2}\right)$.

where range $=$ largest $c_i^{(t)} = \max_{x \in K} |b_i - a_i \cdot x^{(t)}| =: \text{width}(LP)$