

Approximation Algorithms

Suppose you show your problem P is NP-hard. What now?

1. Learn more about inputs
2. Heuristic
3. Approximation algorithm $\leftarrow$ today

Def: For a minimization problem P ,
an algorithm A is an α -approximation algorithm
if for all instances I of P ,

$$A(I) \leq \alpha \cdot \text{OPT} \quad (\alpha \geq 1)$$

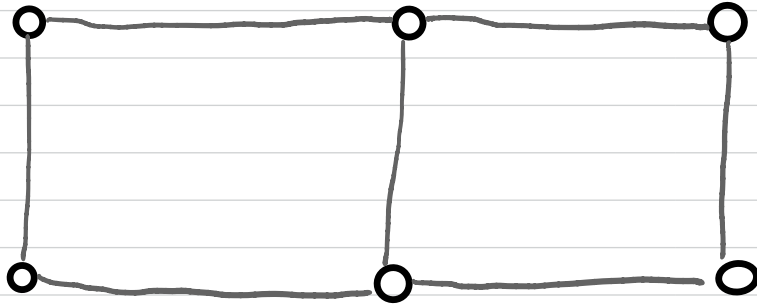
$$(\text{maximization:}) \quad A(I) \geq \alpha \cdot \text{OPT} \quad (0 \leq \alpha \leq 1)$$

Vertex Cover

Input: Graph $G=(V,E)$

Solution: A **vertex cover** $C \subseteq V$ of minimal size

Def: **Vertex Cover** = set of vertices s.t. every edge (u,v) is incident to one of them



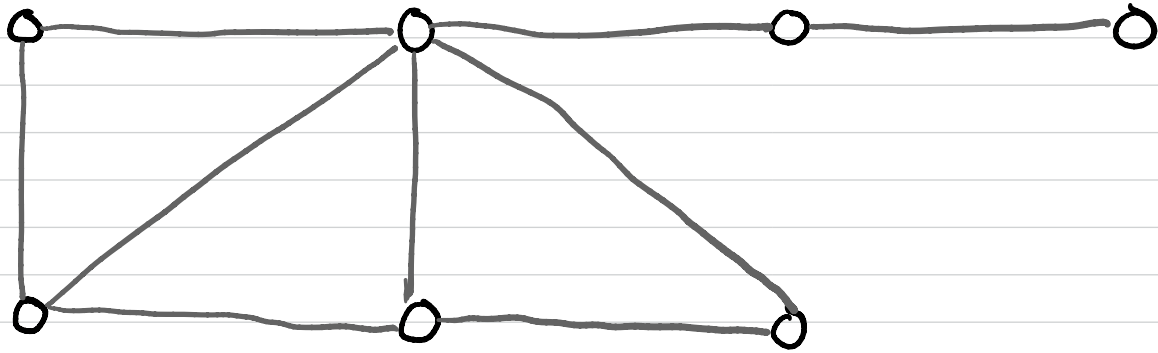
Thm: Vertex Cover is NP-hard (**prove at home**)

Algorithm #1

1. Compute a maximal matching M in G

Def: A **matching** is a set of edges w/ no overlapping vertices.
It is **maximal** if no more edges can be added to it.

Can compute by adding edges **greedily** to M .



2. Output $C = \{ \text{Both endpoints of all edges in } M \}$

Thm: C is a vertex cover and $|C| \leq 2 \cdot \text{OPT}$.

Claim: C is a vertex cover.

Pf: AFSOC that C is **not** a vertex cover.

Then exists $u \text{ --- } v \in E$ s.t. $u, v \notin C$.

Then could add (u, v) to M . But M is maximal! Contradiction. $\square$

Claim: $|C| \leq 2 \cdot \text{OPT}$, $\text{OPT} =$ size of minimum vertex cover.

Pf: Any vertex cover must cover each edge $(u, v) \in M$
 $\Rightarrow$ must include either u or v (or both)

$$\therefore \text{OPT} \geq |M|$$

$$\text{So } |C| = 2 \cdot |M| \leq 2 \cdot \text{OPT}. \quad \square$$

Algorithm # 2: LP

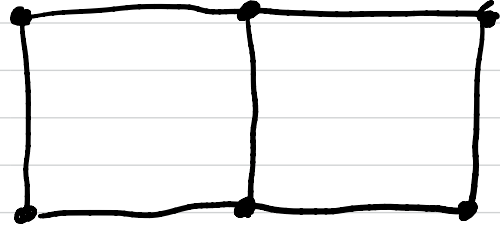
Variable x_i for each vertex i

(ideally) $x_i = \begin{cases} 1 & \text{if } i \in \text{Vertex cover} \\ 0 & \text{otherwise} \end{cases}$

Constraints: $0 \leq x_i \leq 1$

$$x_i + x_j \geq 1 \quad \forall \text{ } i \text{ --- } j \in E$$

Objective: $\min \sum_i x_i$



Claim: $LP\text{-}OPT \leq OPT \text{ Vertex Cover}$

Pf: OPT Vertex Cover is feasible solution to LP. But also fractional solutions! $\square$

"Alg": 1. Solve LP to get optimal solution $\{x_i^*, i=1, \dots, n\}$.

2. Convert x^* to a real vertex cover. "Rounding"

Rounding the LP

- Let $\{x_i^* \mid i=1 \dots n\}$ be the optimal LP solution
- Rounding rule: $\begin{cases} x_i^* \geq \frac{1}{2} \\ x_i^* < \frac{1}{2} \end{cases} \rightarrow \begin{matrix} \text{include in } C \\ \text{do not include in } C \end{matrix}$

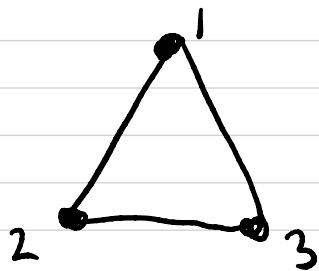
Claim: C is a vertex cover

Pf: $i \text{ --- } j \Rightarrow x_i^* + x_j^* \geq 1 \Rightarrow \text{either } x_i^* \geq \frac{1}{2} \text{ or } x_j^* \geq \frac{1}{2} \Rightarrow i \in C \text{ or } j \in C. \square$

Claim: $|C| \leq 2 \cdot \text{OPT}$.

Pf: $\forall i \in C$, Algorithm pays 1 unit
LP pays $x_i^* \geq \frac{1}{2}$

$\Rightarrow \text{total cost of Alg} = |C| \leq 2 \cdot \text{LP-OPT} \leq 2 \cdot \text{OPT}. \square$



$$\min X_1 + X_2 + X_3$$

$$\text{s.t. } X_1 + X_2 \geq 1$$

$$X_2 + X_3 \geq 1$$

$$X_3 + X_1 \geq 1$$

$$0 \leq X_1, X_2, X_3 \leq 1$$

$$X_1^* = X_2^* = X_3^* = \frac{1}{2} \Rightarrow \text{LP-OPT} = X_1^* + X_2^* + X_3^* = \frac{3}{2}$$

$$\text{Optimal Vertex Cover} = |\{1, 2\}| = 2$$

$$\text{Alg output} = |\{1, 2, 3\}| = 3$$

(Metric) Traveling Salesperson Problem

Input:

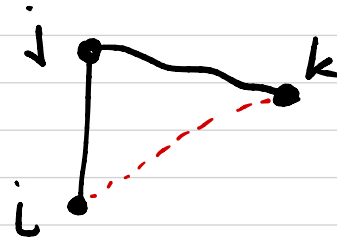
- n cities
- pairwise distance $d_{ij} \forall i \neq j$

Solution: Minimum distance tour
visiting every node exactly once

Metric assumption: triangle inequality

$$\forall i, j, k \quad d_{ij} + d_{jk} \geq d_{ik}$$

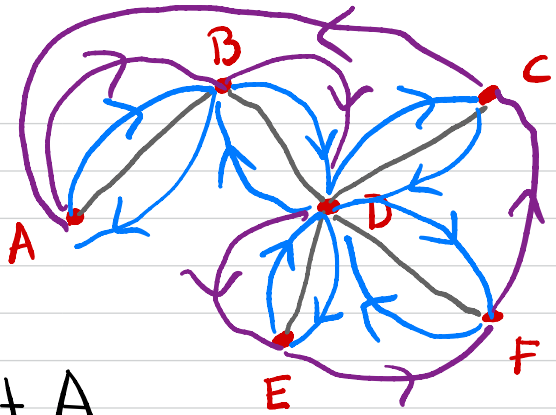
(direct routes are shortest)



Algorithm:

1.) Find the MST T

$$\text{cost}(T) \leq \text{cost}(\text{optimal TSP tour})$$



2.) DFS traversal of T , starting at A

$A \rightarrow B \rightarrow D \rightarrow E \rightarrow D \rightarrow F \rightarrow D \rightarrow C \rightarrow D \rightarrow B \rightarrow A$

$$\text{cost}(\text{DFS traversal}) = 2 \cdot \text{cost}(\text{tree } T)$$

3.) Skip all repeated vertices in the traversal

$A \rightarrow B \rightarrow D \rightarrow E \rightarrow F \rightarrow C \rightarrow A = \text{output}$

$$\text{cost}(\text{TSP tour output}) \leq \text{cost}(\text{DFS traversal})$$

$$\therefore \text{cost}(\text{output}) \leq 2 \cdot \text{OPT}$$